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THE QUOTIENT FIELD AS A TORSION-FREE COVERING MODULE

BY

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ABSTRACT

R will denote a commutative integral domain with quotient field Q. A torsion-free cover of a module M is a torsion-free module F and an R-epimorphism $\sigma : F \to M$ such that given any torsion-free module G and $\lambda \in \mathrm{Hom}_R(G, M)$ there exists $\mu \in \mathrm{Hom}_R(G, F)$ such that $\sigma\mu = \lambda$. It is known that if M is a maximal ideal of R, $R \to R/M$ is a torsion-free cover if and only if R is a maximal valuation ring. Let E denote the injective hull of R/M then $R \to R/M$ extends to a homomorphism $Q \to E$. We give necessary and sufficient conditions for $Q \to E$ to be a torsion-free cover.

Throughout R will denote a commutative integral domain with quotient field Q. For an R-submodule $T \subset Q$, $\pi : Q \to Q/T$ will denote the canonical surjection. The notion of a torsion-free cover is due to Enochs [2]. Enochs [3, cor. 1, p. 51] proved that if M is a maximal ideal of R, $R \to R/M$ is a torsion-free cover if and only if R is a maximal valuation ring. Let $E = E(R/M)$ denote the injective hull of R/M, then $R \to R/M$ can be extended to $Q \to E$. It seems natural to ask when $Q \to E$ is a torsion-free cover. If $Q \to E$ is a torsion-free cover, then, by definition, $Q \to E$ is a surjection. We give necessary and sufficient conditions for $Q \to Q/T$ to be a torsion-free cover where T is an R-submodule of Q. We make frequent use of the results of Matlis [4] and [5] and the construction of the covering module due to Banaschewski [1]. Banaschewski proved that for an R-module A the evaluation map $T(A) \to A$ is a torsion-free cover of A where $T(A) = \{f \mid f : Q \to E(A),\ f(1) \in A\}$.

We begin with a useful lemma.

LEMMA. *If $T \subset Q$ is an R-submodule and some $g \in \mathrm{Hom}_R(Q, Q/T)$ is a torsion-free cover then Q/T is an indecomposable injective R-module.*

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PROOF. From [3, cor. 1, p. 42] Q/T is injective. From Banaschewski's construction $Q \cong \mathrm{Hom}_R(Q, Q/T)$ and [4, prop. 4, p. 575] implies Q/T is indecomposable.

The converse is not rrue. Let R be an almost maximal valuation domain which is not maximal, then Q/T is an indecomposable injective for all R-submodules $T \subset Q$. It will follow from Theorem 2 below that there is some $T \subset Q$ such that $\pi : Q \to Q/T$ is not a torsion-free cover. Our first theorem gives necessary and sufficient conditions for $\pi : Q \to Q/T$ to be a torsion-free cover.

THEOREM 1. *For a proper R-submodule T of Q the following are equivalent:*

(1) *$\pi : Q \to Q/T$ is a torsion-free cover.*

(2) (a) *$\mathrm{Hom}(Q/S, Q/T)$ is naturally isomorphic to $(T:S) = \{q \mid q \in Q, qS \subset T\}$ for all $S \subsetneqq Q$, and*

(b) *Q/T is injective.*

(3) (a) *Every $f \in \mathrm{Hom}(Q, Q/T)$ is a surjection with kernel isomorphic to T, and*

(b) *$\mathrm{Hom}(Q/T, Q/T)$ is naturally isomorphic to $(T:T)$, and*

(c) *Q/T is injective.*

(4) (a) $\mathrm{Ext}^1_R(Q, T) = 0$.

(b) *Q/T is injective.*

PROOF. (1) $\Rightarrow$ (2). Identify $(T:S)$ with the obvious R-submodule of $\mathrm{Hom}(Q, Q)$. If $q \in (T:S)$, q induces an element of $\mathrm{Hom}(Q/S, Q/T)$. It is routine to check that this correspondence is an R-isomorphism. As noted in the lemma, (2b) follows from [3, cor. 1, p. 42].

(2) $\Rightarrow$ (3). $S = 0$ gives (3a) while $S = T$ gives (3b). (3c) is a hypothesis.

(3) $\Rightarrow$ (4). To show $\mathrm{Ext}(Q, T) = 0$ it is sufficient to show that $\mathrm{Hom}_R(Q, Q) \to \mathrm{Hom}_R(Q, Q/T)$ is onto: Let $0 \neq f \in \mathrm{Hom}(Q, Q/T)$ have kernel S. From (3a) $S = qT$ for some $0 \neq q \in Q$. Then $Q \xrightarrow{q} Q \xrightarrow{f} Q/T$ has kernel T and so induces a map $g : Q/T \to Q/T$. From (3b) g is induced by some $q' \in (T:T)$. Thus we have $f = \pi(q'q^{-1})$ as desired.

(4) $\Rightarrow$ (1). Q/T is injective and there is no non-zero pure submodule of Q contained in T. Thus, it is easy to argue that $\pi : Q \to Q/T$ is a torsion-free cover if and only if any diagram

$$\begin{array}{ccc} & Q & \\ \swarrow & & \searrow \\ Q & \xrightarrow{\pi} & Q/T \end{array}$$

can be completed commutatively. Equivalently, $\mathrm{Hom}(Q, Q) \to \mathrm{Hom}(Q, Q/T)$ is onto. This follows from (4a).

Even when covers are known to exist an explicit description of the cover for a particular module is rarely easy. A logical place to begin is with the simple modules as in Enochs [2, lemma 5, p. 887] and [3, theor. 5.1, p. 48] or the cyclic modules as in Banaschewski [1, prop. 6, p. 69]. In each of these cases the covering module is a ring. One is tempted to conjecture, at least for a simple module, that the covering module must be a ring. In our present situation this is so.

LEMMA 2. *If M is a maximal ideal of R and Q/M is the injective hull of R/M then $\psi : \mathrm{Hom}(Q/M, Q/M) \to R/M$ is a torsion-free cover of R/M where $\psi(f) = f(\bar{1})$ for all $f \in \mathrm{Hom}(Q/M, Q/M)$.*

PROOF. Since Q/M is the injective hull of R/M the evaluation map $\xi : T(R/M) \to R/M$ is a torsion-free cover where $T(R/M) = \{f \mid f \in \mathrm{Hom}(Q, Q/M)$ and $f(1) \in R/M\}$ and $\xi(f) = f(1)$. Each $f \in T(R/M)$ induces an $\bar{f} \in \mathrm{Hom}(Q/M, Q/M)$. This correspondence is easily seen to be an R-isomorphism.

COROLLARY. *If M is a maximal ideal of R and $\pi : Q \to Q/M$ is a torsion-free cover then $R \to R/M$ is a torsion-free cover of R/M.*

PROOF. From Lemma 2, $\xi : \mathrm{Hom}(Q/M, Q/M) \to R/M$ is a torsion-free cover of R/M. By Theorem 1, $(M : M) \to R/M : q \to q + M$ is also a torsion-free cover of R/M. Then clearly $(M : M) = R$.

THEOREM 2. *The following statements are equivalent:*

(1) *$\pi : Q \to Q/T$ is a torsion-free cover for all R-submodules $T \subsetneqq Q$.*

(2) *$\pi : Q \to Q/M$ is a torsion-free cover for some maximal ideal M of R.*

(3) *R is a maximal valuation ring.*

PROOF. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). From the previous corollary $R \to R/M$ is a torsion-free cover. By [3, cor. 1, p. 51] R is a maximal valuation ring.

(3) $\Rightarrow$ (1). When R is a maximal valuation ring Q/T is injective and $\mathrm{Ext}(Q, T) = 0$ for every $T \subsetneqq Q$. (1) follows from Theorem 1.

In a similar vein we have:

THEOREM 3. *The following are equivalent:*

(1) *Q is a torsion-free covering module of $E(R/M)$ for every maximal ideal M of R.*

(2) *Any valuation ring V between R and Q is a maximal valuation ring and* $\mathrm{Hom}_{R_M}(V, E(R/M))$ *is the V-injective hull of the unique simple V-module for each valuation ring V between R_M and Q.*

PROOF. (1) $\Rightarrow$ (2). Note that if V is a valuation ring between R and Q, then V contains R_M for some maximal ideal M of R. By assumption, Q covers $E(R/M)$ as an R-module. By [3, prop. 6.1, p. 50] Q covers $E(R/M)$ as an R_M-module. Thus $Q \cong \mathrm{Hom}_{R_M}(Q, E(R/M))$. Apply [4, prop. 6, p. 577].

(2) $\Rightarrow$ (1). From [4, prop. 6, p. 577] we have $Q \cong \mathrm{Hom}_{R_M}(Q, E(R_M/MR_M)) \cong \mathrm{Hom}_R(Q, E(R/M))$ for every maximal ideal M of R. Thus Q covers $E(R/M)$ for every maximal ideal M of R.

Another easy result is

THEOREM 4. *Let R be a valuation ring. $\pi : Q \to Q/R$ is a torsion-free cover if and only if R is a maximal valuation ring (hence complete in the R-topology, which is equivalent to* $\mathrm{Ext}(Q, R) = 0$) ([5, ch. 2]).

In case R is Noetherian we can say more.

THEOREM 5. *Assume R is Noetherian. If $\pi : Q \to Q/T$ is a torsion-free cover for some $T \subsetneqq Q$, then there is a prime ideal P of R such that R_P is a complete, Noetherian, local domain and* $\dim R_P = 1$.

PROOF. There is a prime ideal P of R such that $E(R/P) \cong Q/T$. Since $Q \to E(R/P)$ is a torsion-free cover, $Q \cong \mathrm{Hom}_R(Q, E(R/P)) \cong \mathrm{Hom}_{R_P}(Q, E(R_P/PR_P))$. By [4, prop. 5, p. 575] R_P is complete and $\dim R_P = 1$. Applying [4, theor. 4, p. 578] and [4, theor. 2, p. 572] in order shows that V is a discrete valuation ring finitely generated over R_P. Our Theorem 3 implies that any valuation ring between R_P and Q is maximal.

The example described in [5, p. 113] is pertinent: Let k be a field and x an indeterminate over k. Let R be the subring of the ring of formal power series $k[[x]]$ with the first degree term missing. R is a complete, local, Noetherian domain with $\dim R = 1$ and maximal ideal (x^2, x^3) generated by x^2 and x^3. $k[[x]]$ is the integral closure of R in its quotient field $Q = k((x))$. It can be shown that $\pi : Q \to Q/R$ is a torsion-free cover. Thus R need not be a valuation ring for this to hold (cf. Theorem 4). From Theorem 2 we see that $\pi : Q \rightarrow Q/(x^2, x^3)$ is not a torsion-free cover. This ring R also satisfies the hypothesis of Theorem 5.

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